

## Higher-dimensional Temperley–Lieb algebras

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2007 J. Phys. A: Math. Theor. 40 F895

(<http://iopscience.iop.org/1751-8121/40/38/F01>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.144

The article was downloaded on 03/06/2010 at 06:13

Please note that [terms and conditions apply](#).

## FAST TRACK COMMUNICATION

# Higher-dimensional Temperley–Lieb algebras

**Marcos Alvarez and Paul P Martin**

Centre for Mathematical Science, City University, Northampton Square, London EC1V 0HB, UK

E-mail: [m.alvarez@city.ac.uk](mailto:m.alvarez@city.ac.uk) and [p.p.martin@city.ac.uk](mailto:p.p.martin@city.ac.uk)

Received 25 April 2007, in final form 17 July 2007

Published 4 September 2007

Online at [stacks.iop.org/JPhysA/40/F895](http://stacks.iop.org/JPhysA/40/F895)**Abstract**

A category which generalizes to higher dimensions many of the features of the Temperley–Lieb category is introduced.

PACS number:

Mathematics Subject Classification: 81R50, 82B20

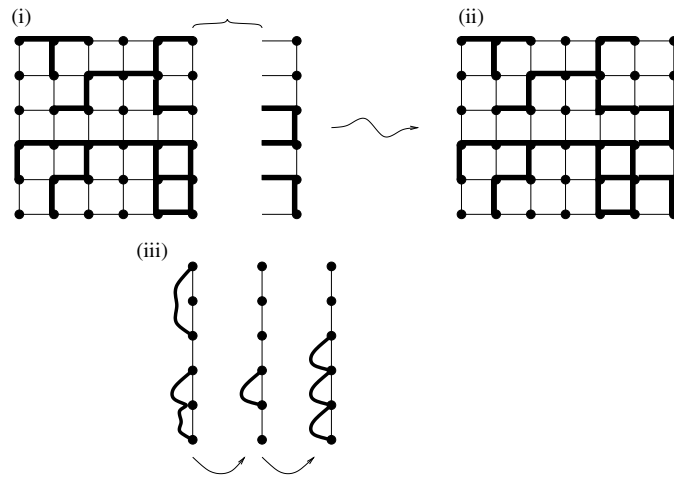
## 1. Introduction

The Temperley–Lieb category provides a useful tool in computational 2D lattice statistical mechanics [5]. Its representation theory is the interlocutor between several different ‘equivalent’ lattice models (Potts, IRF, 6-vertex, etc) [5, 21], is universal among an important class of solutions to integrability conditions and provides the invariant theory for  $U_q sl_2$  [13, 15]. Its categorical structure means that its representation theory can be analysed in great generality [1, 21] (it is universal among monoidal dual categories with certain natural properties, and was the starting point for Khovanov homology [17]). It is also important in a number of other areas of mathematics and physics [16, 19].

In this communication we define a large family of algebras (in fact, categories) that includes the Temperley–Lieb algebra as a particular case. Our construction is based on a notion of diagrams as topological objects embedded in a Euclidean space of arbitrary dimension. The case of dimension 2 then corresponds to Temperley–Lieb diagrams. Before presenting our construction, we will briefly review its physical motivation and the type of physical situations in which we anticipate it will be of relevance.

From a physical point of view, it is natural to expect a three-dimensional analogue of the Temperley–Lieb algebra to play a significant part in the formulation of lattice gauge theories. The role of the lines in Temperley–Lieb diagrams would then be played by propagating Wilson loops, whose evolution in the lattice would define a system of surfaces in three dimensions.

More precisely, the physical setting for the Temperley–Lieb algebra from which we want to generalize is as follows [4, 8, 21]. For  $G$  a graph with vertex set  $V_G$  and edge set  $E_G$  we



**Figure 1.** (i) A subgraph of a square lattice and an extra layer. (ii) The corresponding new subgraph. (iii) A sequence showing: the connectivity of the original subgraph (running # = 12); the connectivity after adding the new horizontal edges (running # = 12 + 3); the connectivity after adding the new vertical edges (running # = 12 + 3 - 2).

associate a  $Q$ -state Potts variable  $\sigma_i \in \underline{Q} := \{1, 2, \dots, Q\}$  with each  $i \in V_G$ . One starts with the Potts Hamiltonian for  $G$ ,

$$H_G = \sum_{(ij) \in E_G} \delta_{\sigma_i, \sigma_j} + h \sum_{i \in V_G} \delta_{\sigma_i, 1}.$$

We take the magnetic field parameter  $h = 0$  and form the partition function

$$Z_G(\beta) = \sum_{\{\sigma_i\}} \exp(\beta H_G) = \sum_{\{\sigma_i\}} \prod_{(ij) \in E_G} \exp(\beta \delta_{\sigma_i, \sigma_j}) = \sum_{\{\sigma_i\}} \prod_{(ij) \in E_G} (1 + v \delta_{\sigma_i, \sigma_j})$$

where  $v = \exp(\beta) - 1$ . Expanding we have

$$Z_G(\beta) = \sum_{\{\sigma_i\}} \sum_{G' \in \mathcal{P}(E_G)} \prod_{(ij) \in G'} v \delta_{\sigma_i, \sigma_j} = \sum_{G' \in \mathcal{P}(E_G)} v^{|G'|} Q^{\#(G')} \tag{1.1}$$

where  $|G'|$  is the number of edges and  $\#(G')$  is the number of connected components of  $G'$  regarded as a subgraph of  $G$  in the obvious way. For example, figure 1(i) shows a subgraph  $G'$  on a square lattice, with  $\#(G') = 12$ .

We can now consider the RHS of (1.1) in its own right (as a ‘dichromatic’ polynomial in variables  $v$  and  $Q$ ). The exercise is to construct a transfer matrix formulation in which to compute it, in cases where  $G$  has (‘time’) translation symmetry. We also require that  $G$  embeds in some Euclidean space and that its edges (and hence the Potts interactions) are local. However, even this is not enough to make the interactions in the dichromatic polynomial formulation local, since  $\#(G')$  is not local. Instead we need to introduce an entirely different state space. Although the restriction is not necessary, for the sake of simplicity we will describe this by considering the example of the  $n$ -site wide square lattice.

In adding an extra layer to this lattice we are adding  $2n - 1$  edges. As ever in a transfer matrix formalism, the problem is to find a set of states which keep enough information about the old lattice  $G$  to determine  $\#(G')$  for the new one. It will be evident that each state must record which of the last layer of vertices in  $G$  are connected to each other (by some route in  $G$ —cf figures 1(i)–(iii)). Neither the details of the connecting routes nor any other information is

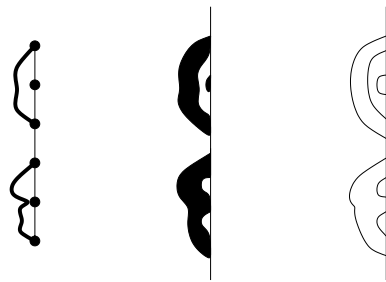


Figure 2. Temperley–Lieb diagram.

needed; thus our state set is simply contained in the set of partitions of the last layer of vertices. It is straightforward to see that (in the square, or otherwise plane, lattice case) precisely the set of ‘plane’ partitions are needed. These are the partitions realizable by noncrossing paths in the interior when the vertices are arranged around the edge of a disc.

Pictures of such paths are called Whitney diagrams [21]. If instead we represent plane partitions by boundaries of connected regions, these diagrams become Temperley–Lieb (or boundary) diagrams on the disc. Note that these are plane *pair* partitions of double the number of vertices. See figure 2 for an example. Note that the original lattice itself has all but disappeared from the state space (replaced by a topological/combinatorial construct).

Finally we note that in order to compute correlation functions some further information must be retained (essentially the details of connections also with the vertices on the left-hand side of the graph in figure 1). This corresponds to Temperley–Lieb diagrams on the rectangle—i.e. with both in-vertices and out-vertices. These diagrams may be composed by juxtaposition at one edge of the rectangle when the number of states agrees. With an appropriate reduction rule for interior loops (replace by a factor  $\sqrt{Q}$ ) this becomes the Temperley–Lieb algebra (indeed category, indeed monoidal category).

Casting the state space in this form is certainly beautiful and computationally convenient (see [23]), but it is not the same as integrability. Since the Potts model is integrable under certain conditions, solutions to the Yang–Baxter equations can be constructed using Temperley–Lieb diagrams, but such exercises will not be our focus in the present communication.

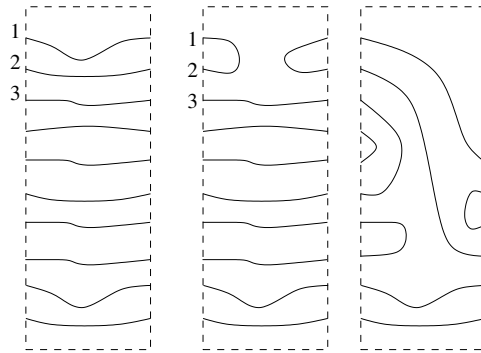
The following set of Temperley–Lieb diagrams generate the Temperley–Lieb algebra on  $n$  vertices (i.e.  $n$  in- and  $n$  out-vertices). The identity diagram is the rectangle in which each in-vertex is connected to the corresponding out-vertex. The diagram  $D_i$  is like the unit except that in-vertices  $i$  and  $i + 1$  are connected, and out-vertices  $i$  and  $i + 1$  are connected. (See figure 3. Here the vertices are attached to the vertical edges of the diagram. We will later adopt the convention that the vertices are attached to the horizontal edges instead.) The generators are  $D_1, \dots, D_{n-1}$ . The state space we have constructed induces a representation of these elements. The transfer matrix is then

$$T = \prod_i \left( 1 + \frac{v}{\sqrt{Q}} R(D_{2i}) \right) \prod_i \left( \frac{v}{\sqrt{Q}} + R(D_{2i-1}) \right) \quad (1.2)$$

and

$$Z(\beta) = \text{Tr}(T^n).$$

Finally, the trace can be decomposed into the irreducible representations in  $R$  (amongst other partial diagonalizations). The close relationship this engenders between representation theory and correlation functions (see, e.g., [23]) is what we aim to generalize.



**Figure 3.** Temperley–Lieb identity diagram, diagram  $D_1$ , and a diagram with different numbers of in and out-vertices.

Both mathematically and physically it is convenient to assemble the Temperley–Lieb algebras into the Temperley–Lieb category (diagrams with different numbers of in- and out-vertices [21]; see later for details). Accordingly we approach the problem of generalization here by casting the problem in a categorical framework, and defining a number of categories generalizing this category. The claim is that this is natural, and ultimately makes the exposition more efficient.

The generalization to ‘higher dimensions’ (in the mean-field sense of every lattice point being a neighbour) is the partition category [22], but it is natural also to seek a 3D version. A 3D lattice subalgebra of the partition algebra was studied by Dasmahapatra and Martin [10] but it has very few of the beautiful properties of the Temperley–Lieb and partition categories (cf Baxter–Bazhanov tetrahedra for example [6, 7, 14]), and essentially no progress has been made in this direction. Here we describe a generalization of the full Temperley–Lieb category to 3D that is closer in spirit to the cobordism category of topological quantum field theory [3, 20] and to lattice gauge theories [21, p 278]. The problem with such a construction is that there are significant initial problems with well-definedness, but the payoff is potential access to generalizations of several of the structures mentioned above. In this communication we solve the well-definedness problem, and the core enumeration problem in using the resultant algebras. Applications will be discussed elsewhere.

## 2. Concrete diagram categories

We begin by selecting a direction in  $d$ -dimensional Euclidean space  $\mathbb{E}^d$  which we call ‘time’, coordinatized by a real number  $t$ . For fixed  $d \in \mathbb{N}$  and  $t \geq 0$  we define  $E_t = \mathbb{R}^{d-1} \times [0, t]$ . If  $D$  and  $E$  are subsets of  $\mathbb{E}^d$ , we write  $C_E(D)$  for the number of connected components of  $E \setminus D$  in the Euclidean topology.

**Definition 1.** *The set  $S^{d,t}$  of concrete diagrams of duration  $t$  is the set of compact subsets of  $E_t$  such that  $D \in S^{d,t}$  if and only if, for all  $D'$  obtained from  $D$  by removing a single point, we have  $C_{E_t}(D') = C_{E_t}(D) - 1$ . Then  $E_t$  is called a universe for  $D$ . Write  $S^d$  for  $\cup_{t \geq 0} S^{d,t}$  and  $S_{\square}^{d-1}$  for the set of concrete diagrams in  $S^d$  with  $t = 0$ .*

For  $d = 2$ , concrete diagrams coincide with concrete Temperley–Lieb diagrams, and in this sense our construction provides a ‘higher-dimensional’ generalization. For  $d = 3$ , a concrete diagram can be thought of as consisting of a number of embedded submanifolds

whose boundaries all lie on either of two parallel planes of constant time. See figures 1 and 2 for examples.

A *component* of  $D \in S^{d,t}$  is any point  $x \in D$  together with the maximal subset  $X \ni x$  such that  $C_{E_t}(D \setminus X) = C_{E_t}(D \setminus \{x\})$ . A *bubble* is a component which does not intersect the boundaries of  $E_t$ . A component  $c$  has a *handle* if it properly contains a non-contractible loop not homologous in  $c$  to a subset of the boundary of  $E_t$ . Write  $|D|$  for the number of components of a concrete diagram  $D$ , and  $b(D)$  for the number of bubbles.

The *number of handles* of a concrete diagram  $D \in S^3$  is the genus of  $D$ , written  $g(D)$ . We will write  $\chi(D)$  for the Euler number of  $D$  [2, 26].

**Definition 2.** Let  $F, F' \subset \mathbb{R}^{d-1}$ . Then  $S^{d,t}[F, F']$  is the subset of  $S^{d,t}$  of concrete diagrams which intersects the  $t$ -boundary of  $E_t$  in  $F$  and the 0-boundary in  $F'$ . Set  $S^d[F, F'] = \cup_{t \geq 0} S^{d,t}[F, F']$ . The boundary configuration of  $D \in S^d[F, F']$  is the ordered pair  $(F, F')$ .

(Note that  $S^d[F, F'] = \emptyset$  unless  $F, F' \in S_{\square}^{d-1}$ . In practice, we shall restrict attention to cases in which  $F$  and  $F'$  are unions of  $(d - 2)$ -spheres.)

For  $T$  a set, let  $\wp(T)$  be the set of partitions of  $T$  and  $\mathcal{P}(T)$  the power set of  $T$ . For  $U$  and  $V$  sets and  $u \in \wp(U), v \in \wp(V)$  we write  $u * v$  for the partition algebra composition of  $u$  with  $v$  [25, p 158], or more generally the *Ag* product in [22, p 77].

For  $D \in S^d[F, F']$  define its *connectivity*  $p(D)$  as the partition of the components (with respect to the obvious extension of the  $\mathbb{E}^{d-1}$  topology) of  $F \dot{\cup} F'$ , such that  $a, b$  are in the same part if they are in the same component of  $D$ . For example, for  $D$  the concrete diagram with labelled boundaries in figure 2, the connectivity is  $p(D) = \{\{1, 1'\}, \{2\}, \{2'\}\}$ .

For fixed  $d \in \mathbb{N}$  define the map  $R : S^d \rightarrow S^d$  by  $R(D)$  being  $D$  with all bubbles removed. Define  $S_{\min}^d \subset S^d$  as the subset of elements with no bubbles and no handles. The elements of  $S_{\min}^d$  are called *minimal*. For example, all the diagrams shown in figure 2 are minimal, whereas the diagram of figure 1 is not, as it contains a bubble.

For any two sets  $A$  and  $B$ , define the symmetric difference  $A \wedge B := (A \cup B) \setminus (A \cap B)$ . For  $A, B \in S^d$  we write  $A \overline{\wedge} B$  for the closure of  $A \wedge B$  in the Euclidean metric topology. Two concrete diagrams  $D$  and  $D'$  are  $\wedge$ -composable if they have a universe in common and  $D \cap D'$  is a finite union of disjoint closed  $(d - 1)$ -balls. It can then be shown that [24]

**Lemma 3.** If  $D$  and  $D'$  are  $\wedge$ -composable then  $D \overline{\wedge} D' \in S^d$ .

For  $B \in S^d$ , define  $\text{dom } B$  as the set of all  $A \in S^d$  which are  $\wedge$ -composable with  $B$ . Then we define the map  $\delta_B : \text{dom } B \rightarrow S^d$  by  $A \mapsto A \overline{\wedge} B$ .

For any point  $P \in \mathbb{E}^d$  with time coordinate  $t_P$ , the *time translate*  $P_\tau$  is the point in  $\mathbb{E}^d$  with the same projection on the  $t = 0$  subspace as  $P$  but with time coordinate  $t_P + \tau$ . For any  $Z \subset \mathbb{E}^d$ , we define  $Z_\tau$  by  $P \in Z_\tau \Leftrightarrow P_{-\tau} \in Z$ . The following diagram composition lemma follows now from definition 1.

**Lemma 4.** If  $A \in S^{d,t}[F', F]$  and  $B \in S^{d,\tau}[F, F'']$ , then  $A \circ B := A_\tau \cup B$  is in  $S^d[F', F'']$ .

Let us now define the triple  $\mathcal{C}^d = (S_{\square}^{d-1}, \text{hom}(-, -), \circ)$  consisting of the ‘object set’  $S_{\square}^{d-1}$ ; and for each pair of objects  $E, F \in S_{\square}^{d-1}$  the collection of ‘morphisms’  $\text{hom}(E, F) = S^d[E, F]$ ; and composition of morphisms defined by  $\circ$ -composition of concrete diagrams. The morphism in  $\text{hom}(F, F)$  of duration one whose sections of constant time are time translates of  $F$  will be denoted  $\mathcal{I}_F$ .

**Theorem 5.**  $\mathcal{C}^d$  is a category.

**Proof.** We require to show associativity of  $\circ$ , and existence of identity element in each  $\text{hom}(F, F)$ . The former is clear, the latter is the concrete diagram of duration zero.  $\square$

Let  $A \in S^3[F', F]$  and  $B \in S^3[F, F'']$ . Then

$$\chi(A \circ B) = \chi(A) + \chi(B), \quad (2.1)$$

(which follows directly from the definition of  $\chi(A)$  as an alternating sum of Betti numbers, or as a sum of Morse indices [27], of  $A$ ) and

$$g(A \circ B) = g(A) + g(B) + |A \circ B| - |A| - |B| + |F|. \quad (2.2)$$

We will later need other results that follow easily from equation (2.1) which we give without proof. Here we assume that all  $\circ$ -compositions are defined;  $D$  is a minimal concrete diagram with the same connectivity as  $A \circ B$ , and  $G$  is a minimal concrete diagram with the same connectivity as  $B \circ C$ :

$$\begin{aligned} g(A \circ B) + g(D \circ C) &= g(B \circ C) + g(A \circ G), \\ b(A \circ B) + b(D \circ C) &= b(B \circ C) + b(A \circ G). \end{aligned} \quad (2.3)$$

### 3. Equivalence relations on $S^3$

In this section we set  $d = 3$ . We will write  $i$  for the usual relation of isotopy (i.e., a continuous, one-parameter family of homeomorphisms [28]). If  $j$  is a specific isotopy with parameter  $s \in [0, 1]$  and  $A$  is a concrete diagram, we write  $j_u(A)$  for the image of  $A$  under  $j$  at  $s = u$ . In particular,  $j_0(A) = A$  for all  $A$ . Two concrete diagrams  $A$  and  $B$  are *isotopic* iff there is an isotopy  $j$  such that  $j_s(A)$  is a concrete diagram for all  $s$ , and  $j_1(A) = B$ .

This defines the isotopy relation  $i$  in the set of concrete diagrams. It is obviously an equivalence relation.

Let  $A$  and  $B$  be concrete diagrams. If  $A$  and  $B$  are isotopic by an isotopy  $j$  such that the boundaries of  $j_s(A)$  are time translates of the boundaries of  $A$  for all  $s$  we say that they are *strongly isotopic*.

For example, the isotopy whose action on a point in  $\mathbb{E}^3$  with coordinates  $(x, y, t)$  is  $j_s(x, y, t) = (x, y, (1+s)t)$  is strong. Strong isotopy is clearly an equivalence relation, which we denote  $\mathfrak{si}$ .

#### Definition 6.

- (1) For any  $A, B \in S^3$ , define relation  $\tau_1$  by  $A\tau_1 B$  if there is a torus  $T$  such that  $A \cap T$  is a disc, and  $B = \delta_T A$ . Relation  $\tau$  is the transitive closure of  $\tau_1$ .
- (2) Relation  $\mathfrak{sh}$ , called strong heterotopy, is the reflexive, symmetric and transitive closure of  $\tau_1$  and  $\mathfrak{si}$ .
- (3) Relation  $\mathfrak{h}$ , called heterotopy, is the reflexive, symmetric and transitive closure of  $\tau_1$  and  $i$ .

The cosets  $S^3[F, F']/\mathfrak{sh}$  and  $S^2[F, F']/i$  (Temperley–Lieb diagrams) are infinite owing to the possible presence of bubbles or handles. But  $S_{\min}^2[F, F']/i$  is finite. We will show in section 4 that  $S_{\min}^3[F, F']/\mathfrak{sh}$  is finite.

The following lemma follows immediately from the previous definitions.

**Lemma 7.** If  $A\mathfrak{h}A'$  then  $b(A) = b(A')$ . If  $A\mathfrak{sh}A'$  then  $p(A) = p(A')$ .

**Definition 8.** For  $D \in S_{\min}^3$ , write  $[D]_{\mathfrak{h}}$  for the restriction of the  $\mathfrak{h}$ -class of  $D$  to  $S_{\min}^3$ , i.e.,  $[D]_{\mathfrak{h}} = \{C \in S_{\min}^3 \mid C\mathfrak{h}D\}$ . Write  $S_{\mathfrak{h}}^3$  for the set of  $\mathfrak{h}$ -classes in  $S_{\min}^3$  and  $S_{\mathfrak{h}}^3[F, F']$  for the

set of  $\mathfrak{h}$ -classes in  $S_{\min}^3[F, F']$ . Similarly, write  $[D]_{\mathfrak{sh}}$  for the restriction of the  $\mathfrak{sh}$ -class of  $D$  to  $S_{\min}^3$ , i.e.,  $[D]_{\mathfrak{sh}} = \{C \in S_{\min}^3 \mid C \mathfrak{sh} D\}$ , and  $S_{\mathfrak{sh}}^3$  for the set of  $\mathfrak{sh}$ -classes in  $S_{\min}^3$ , and  $S_{\mathfrak{sh}}^3[F, F']$  for the set of  $\mathfrak{sh}$ -classes in  $S_{\min}^3[F, F']$ .

By the handle decomposition theorem [27],

**Proposition 9.** For every  $A \in S^3$  there exists  $D \in S_{\min}^3$  such that  $D \mathfrak{tr} R(A)$ .

**Definition 10.** Define  $\mathbb{D}_r : S^3 \rightarrow \mathcal{P}(S_{\min}^3)$  by  $A \mapsto \{D \in S_{\min}^3 \mid D \mathfrak{tr} R(A)\}$ .

The following proposition follows from proposition 9 and definition 6.

**Proposition 2.** For each  $A \in S^3$  there is  $D \in S_{\min}^3$  such that  $\mathbb{D}_r(A) \subseteq [D]_{\mathfrak{sh}}$ . Moreover, if  $\mathbb{D}_r(A) \subseteq [D]_{\mathfrak{sh}}$  and  $\mathbb{D}_r(A) \subseteq [D']_{\mathfrak{sh}}$  then  $[D]_{\mathfrak{sh}} = [D']_{\mathfrak{sh}}$ .

Define  $\mathbb{D}_{\mathfrak{h}}(A)$  and  $\mathbb{D}_{\mathfrak{sh}}(A)$  as, respectively, the  $\mathfrak{h}$ - and  $\mathfrak{sh}$ -class in  $S_{\min}^3$  containing  $\mathbb{D}_r(A)$ .

**Definition 12.** For  $\mathcal{K}$  a ring and  $p, q \in \mathcal{K}$  define the reduction maps  $\mu$  and  $\nu$  as

$$\begin{aligned} \mu : S^3 &\rightarrow \mathcal{K}S_{\mathfrak{h}}^3 & \nu : S^3 &\rightarrow \mathcal{K}S_{\mathfrak{sh}}^3 \\ A &\mapsto p^{g(A)}q^{b(A)}\mathbb{D}_{\mathfrak{h}}(A) & A &\mapsto p^{g(A)}q^{b(A)}\mathbb{D}_{\mathfrak{sh}}(A) \end{aligned}$$

and extend linearly to  $\mathcal{K}S^3$  in each case.

If  $A$  and  $B$  are  $\circ$ -composable, then any  $A' \mathfrak{sh} A$  and  $B' \mathfrak{sh} B$  are also  $\circ$ -composable. The next lemma follows from noticing that the two strong heterotopies relating  $A$  to  $A'$  and  $B$  to  $B'$  combine into a single one that relates  $A \circ B$  to  $A' \circ B'$ .

**Lemma 13.** Let  $A$  and  $B$  be  $\circ$ -composable. Let  $A' \mathfrak{sh} A$  and  $B' \mathfrak{sh} B$ . Then  $(A \circ B) \mathfrak{sh} (A' \circ B')$ .

Therefore we can extend  $\circ$ -composability to  $\mathfrak{sh}$ -classes in a well-defined way by  $\circ$ -composing representatives.

**Definition 14.** Let  $A, B$  be  $\circ$ -composable. Then  $[A]_{\mathfrak{sh}} \bullet [B]_{\mathfrak{sh}} := \nu(A \circ B)$ .

**Theorem 15.** The triple  $C_{sh} = (S_{\mathfrak{sh}}^2, \mathcal{K}S_{\mathfrak{sh}}^3[-, -], \bullet)$  is a category whose morphisms are  $\mathfrak{sh}$ -classes of concrete diagrams.

**Proof.** (i) Associativity of  $\bullet$ -composition of  $\mathfrak{sh}$ -classes follows from associativity of  $\circ$ -composition of concrete diagrams. (ii) The unit in  $S_{\mathfrak{sh}}^3[F, F]$  is the  $\mathfrak{sh}$ -class of the diagram of zero duration in  $S^3[F, F]$ .  $\square$

**Proposition 16.** Let  $A, B \in S^d[F, F']$  and  $A \mathfrak{h} B$ . Then there exist  $L \in S_{\min}^3[F, F]$  and  $R \in S_{\min}^3[F', F']$  such that  $L \mathfrak{h} \mathfrak{J}_F$ ,  $R \mathfrak{h} \mathfrak{J}_{F'}$  and  $A \mathfrak{sh} (L \circ B \circ R)$ .

**Proof.** If  $A \mathfrak{sh} B$  then take  $L = \mathfrak{J}_F$  and  $R = \mathfrak{J}_{F'}$ . Otherwise the  $\mathfrak{h}$ -relation between  $A$  and  $B$  contains isotopies in a neighbourhood of the boundary of  $A$ . Extend those isotopies to a neighbourhood of  $\mathfrak{J}_F$  and  $\mathfrak{J}_{F'}$  in  $\mathfrak{J}_F \circ A \circ \mathfrak{J}_{F'}$  to obtain  $L \circ B \circ R$ . The  $L$  and  $R$  so defined are clearly  $\mathfrak{h}$ -related to  $\mathfrak{J}_F$  and  $\mathfrak{J}_{F'}$ , respectively (in fact, isotopic).  $\square$

By a routine check that the group axioms are satisfied, we have

**Proposition 17.** The  $\mathfrak{sh}$ -classes inside  $[\mathfrak{J}_F]_{\mathfrak{h}}$  in  $S_{\min}^3[F, F]$  form a group  $\Pi_F$  under  $\bullet$ -composition, with unit  $[\mathfrak{J}_F]_{\mathfrak{sh}}$ .

**Definition 18.** For a finite  $X \subset S_{\min}^d$ , define  $\sigma_X = \sum_{A \in X} A \in \mathcal{K}S_{\min}^d$ .



#### 4. $\mathfrak{h}$ -classes and connectivity

In this section we show that the  $S^3[F, F']/\mathfrak{sh}$  is finite.

**Definition 19.** Let  $c$  and  $d$  be components of  $A \in S^d$ . Then  $c$  and  $d$  are neighbours if there is a path connecting  $c$  to  $d$  not intersecting any other component of  $A$ . Let  $c$  and  $d$  be neighbours in  $A$  with path  $P$ , and  $e$  a surface obtained from  $c$  and  $d$  by removing a disc from each and joining the edges with a cylindrical thickening of  $P$  not intersecting any other component of  $A$ . Then  $e$  is a bridging of  $c$  and  $d$  and the cylinder is a bridge connecting  $c$  and  $d$ .

**Theorem 20.** Let  $A, B \in S_{\min}^3[F, F']$ . Then  $A \mathfrak{sh} B$  iff  $p(A) = p(B)$ .

Necessity follows from the definition of  $\mathfrak{sh}$ . A complete proof of sufficiency will be given in [24]. Here we present those ideas of the proof that are relevant to understanding the rest of this communication. The key observation is that any two concrete diagrams in  $S_{\min}^3[F, F']$  with  $|F| + |F'|$  components are necessarily  $\mathfrak{sh}$ -related (in fact,  $\mathfrak{si}$ -related) and have equal connectivities. (Both statements follow from the fact that all components in any such concrete diagram are discs, each disc bounded by exactly one loop in either  $F$  or  $F'$ .) The proof then proceeds by induction in  $k = |F| + |F'| - n$ , where  $n$  is the number of components of  $A$  and  $B$ . Given  $A, B \in S_{\min}^3[F, F']$  with  $p(A) = p(B)$  and  $n < |F| + |F'|$  components (i.e.,  $k > 0$ ), concrete diagrams  $A', B' \in S_{\min}^3[F, F']$  are constructed such that  $p(A') = p(B')$ , and both have  $n + 1$  components (i.e.,  $k$  reduced by 1). The induction hypothesis is that the theorem is true for  $k - 1$ , so that  $A' \mathfrak{sh} B'$ . It is then shown that  $A$  and  $B$  can be reconstructed from  $A'$  and  $B'$  by bridging, in a way that shows that  $A$  and  $B$  are also  $\mathfrak{sh}$ -related. This shows that the theorem is true for  $k$  if it is true for  $k - 1$ . Being true for  $k = 0$  (concrete diagrams in  $S_{\min}^3[F, F']$  with  $|F| + |F'|$  components), it follows that it is always true.

**Proposition 21.** For any  $F, \Pi_F$  is a finite group.

**Proof.** It was established in proposition 17 that  $\Pi_F$  is a group. Because  $F$  is always a finite set, the partition set  $\wp(F \dot{\cup} F)$  is also finite. Hence, by theorem 20, there is a finite number of  $\mathfrak{sh}$ -classes in  $[\mathcal{J}_F]_{\mathfrak{h}}$ .  $\square$

#### 5. Composition of $\mathfrak{h}$ -classes

Consequent to proposition 16, there are fewer  $\mathfrak{h}$ -classes than  $\mathfrak{sh}$ -classes in  $S_{\min}^3[F, F']$ . We regard the extra symmetry  $\mathfrak{h}/\mathfrak{sh}$  as the higher-dimensional counterpart of order-preserving displacements of the endpoints of Temperley–Lieb diagrams along the upper or lower edges. For that reason we regard  $\mathfrak{h}$ -classes as the natural generalization of the notion of Temperley–Lieb diagram to higher dimensions, and they will be the main object of study in the rest of this work.

Our goal, then, is to define a composition of  $\mathfrak{h}$ -classes by which each  $S_{\mathfrak{h}}^3[F, F]$  becomes a finite-dimensional, associative algebra with unit.

**Proposition 22.** Let  $A, A', B, B' \in S_{\min}^3$  and  $A \mathfrak{sh} A'$  and  $B \mathfrak{sh} B'$ . If  $A$  and  $B$  are  $\circ$ -composable, then  $\mu(A \circ B) = \mu(A' \circ B')$ .

**Proof.** From lemma 13,  $\mathbb{D}(A \circ B) = \mathbb{D}(A' \circ B')$ . Therefore we only need to prove that  $b(A \circ B) = b(A' \circ B')$  and  $g(A \circ B) = g(A' \circ B')$ . By lemma 13 we know that  $(A \circ B) \mathfrak{sh} (A' \circ B')$ , and then by lemma 7  $b(A \circ B) = b(A' \circ B')$ . By theorem 20 we know that  $p(A) = p(A')$  and  $p(B) = p(B')$ , hence  $p(A \circ B) = p(A' \circ B')$  and, in particular,  $A \circ B$  and  $A' \circ B'$  have

an equal number of non-bubble components. Hence  $|A \circ B| = |A' \circ B'|$ . But it follows from equation (2.2) that  $g(A \circ B)$  depends only on the number of components of  $A \circ B$  if  $A$  and  $B$  are minimal. Therefore  $g(A \circ B) = g(A' \circ B')$ .  $\square$

This proposition can be extended to show that if  $A_i, A'_i \in S_{\min}^3[F_i, F_{i+1}]$  with  $A_i \mathfrak{sh} A'_i$  for  $i = 1, \dots, n$ , then

$$\mu(A_1 \circ \dots \circ A_n) = \mu(A'_1 \circ \dots \circ A'_n). \tag{5.1}$$

**Definition 23.** A subset of  $[\mathfrak{J}_F]_{\mathfrak{h}}$  is complete if it consists of exactly one element from each  $\mathfrak{sh}$ -class.

**Lemma 24.** Let  $A, A' \in S_{\min}^d[F', F]$  and  $B, B' \in S_{\min}^d[F, F'']$ , with  $A \mathfrak{h} A'$  and  $B \mathfrak{h} B'$ . Let  $X, X'$  be complete subsets of  $[\mathfrak{J}_F]_{\mathfrak{h}}$ . Then

$$\mu(A \circ \sigma_X \circ B) = \mu(A' \circ \sigma_{X'} \circ B').$$

**Proof.** By proposition 21,  $X$  and  $X'$  are finite sets and  $\sigma_X$  and  $\sigma_{X'}$  are defined. By proposition 16 there are  $L_A \mathfrak{h} \mathfrak{J}_{F'}, R_A \mathfrak{h} \mathfrak{J}_F$  such that  $A \mathfrak{sh} (L_A \circ A' \circ R_A)$ , and similarly  $L_B \mathfrak{h} \mathfrak{J}_F, R_B \mathfrak{h} \mathfrak{J}_{F''}$  such that  $B \mathfrak{sh} (L_B \circ B' \circ R_B)$ . Then, for each  $Y \in \sigma_X$ , equation (5.1) gives  $\mu(A \circ Y \circ B) = \mu(L_A \circ A' \circ R_A \circ Y \circ L_B \circ B' \circ R_B)$ . But, by completeness of  $X$  and proposition 17 there exists exactly one  $Y' \in X'$  such that  $(R_A \circ Y \circ L_B) \mathfrak{sh} Y'$ , so  $\mu(A \circ Y \circ B) = \mu(L_A \circ A' \circ Y' \circ B' \circ R_B)$ . Finally,  $(L_A \circ A' \circ Y' \circ B' \circ R_B) \mathfrak{h} (A' \circ Y' \circ B')$  and both have an equal number of bubbles and handles, so  $\mu(A \circ Y \circ B) = \mu(A' \circ Y' \circ B')$ . The lemma follows by summing over  $Y \in X$ .  $\square$

That is,  $\mu(A \circ \sigma_X \circ B)$  depends only on the  $\mathfrak{h}$ -classes of  $A$  and  $B$  and not on the choice of  $X$ . Therefore we have a well-defined composition of  $\mathfrak{h}$ -classes into  $\mathcal{K}S_{\mathfrak{h}}^3$ :

$$[A]_{\mathfrak{h}} \cdot [B]_{\mathfrak{h}} = \mu(A \circ \sigma_X \circ B). \tag{5.2}$$

If  $|X|$  has an inverse in  $\mathcal{K}$  we define a new composition rule that has  $[\mathfrak{J}_F]_{\mathfrak{h}}$  as unit:

$$[A]_{\mathfrak{h}} \cdot [B]_{\mathfrak{h}} = \frac{1}{|X|} \mu(A \circ \sigma_X \circ B). \tag{5.3}$$

As was done in section 3 with  $\mathfrak{sh}$ -classes, the set  $S_{\mathfrak{h}}^3[F, F']$  of  $\mathfrak{h}$ -classes in  $S^3[F, F']$  can be interpreted as the set of morphisms with object set  $S_{\square}^2$  and composition rule given by (5.3). In the next subsection we show that this composition rule is associative. Then,

**Theorem 25.** The triple  $\mathcal{C}_{\mathfrak{h}} = (S_{\square}^2, \mathcal{K}S_{\mathfrak{h}}^3[-, -], \cdot)$  defines a category which will be called the heterotopy category.

The only non-trivial step in the proof of this theorem is associativity:

**Proposition 26.** Let  $[A]_{\mathfrak{h}}, [B]_{\mathfrak{h}}, [C]_{\mathfrak{h}} \in S^3$ . Then

$$([A]_{\mathfrak{h}} \cdot [B]_{\mathfrak{h}}) \cdot [C]_{\mathfrak{h}} = [A]_{\mathfrak{h}} \cdot ([B]_{\mathfrak{h}} \cdot [C]_{\mathfrak{h}}).$$

**Proof.** Let  $X$  be a complete subset of  $[\mathfrak{J}_F]_{\mathfrak{h}}$ . For each  $Y \in X$  define  $[E_Y] = \mathbb{D}(A \circ Y \circ B)$  and  $[H_Y] = \mathbb{D}(B \circ Y' \circ C)$ . Then

$$\begin{aligned} ([A]_{\mathfrak{h}} \cdot [B]_{\mathfrak{h}}) \cdot [C]_{\mathfrak{h}} &= \frac{1}{|X|^2} \sum_{Y, Y' \in X} p^{g(A \circ Y \circ B)} q^{b(A \circ Y \circ B)} p^{g(E_Y \circ Y' \circ C)} q^{b(E_Y \circ Y' \circ C)} \mathbb{D}(E_Y \circ Y' \circ C), \\ [A]_{\mathfrak{h}} \cdot ([B]_{\mathfrak{h}} \cdot [C]_{\mathfrak{h}}) &= \frac{1}{|X|^2} \sum_{Y, Y' \in X} p^{g(A \circ Y \circ H_{Y'})} q^{b(A \circ Y \circ H_{Y'})} p^{g(B \circ Y' \circ C)} q^{b(B \circ Y' \circ C)} \mathbb{D}(A \circ Y \circ H_{Y'}). \end{aligned}$$

Because both  $E_Y \circ Y' \circ C$  and  $A \circ Y \circ H_{Y'}$  are in  $\mathbb{D}(A \circ Y \circ B \circ Y' \circ C)$  we have  $\mathbb{D}(E_Y \circ Y' \circ C) = \mathbb{D}(A \circ Y \circ H_{Y'})$ . Therefore we only need to prove that

$$\begin{aligned} g(A \circ Y \circ B) + g(E_Y \circ Y' \circ C) &= g(A \circ Y \circ H_{Y'}) + g(B \circ Y' \circ C), \\ b(A \circ Y \circ B) + b(E_Y \circ Y' \circ C) &= b(A \circ Y \circ H_{Y'}) + b(B \circ Y' \circ C). \end{aligned}$$

Those equations follow from equations (2.3).  $\square$

Rule (5.3) does not provide a practical means of computing the composition of two given  $\mathfrak{h}$ -classes, except in the simplest cases where representative concrete diagrams can be concatenated by hand. We address this question next.

## 6. Computing the multiplication table

In order to generate examples, and to begin analysing the structure of the algebra, we need to be able to compute compositions efficiently. If we wished to compute the multiplication table for the algebra  $S_{\mathfrak{h}}^3[F, F']$  with  $F$  two concentric circles, we could use as a concrete basis the concrete diagrams in figure 2, concatenate them by hand and finally apply the reduction map  $\mu$ . This would result in the following multiplication table, in which  $D_1, \dots, D_9$  refers to the concrete diagrams shown in figure 2, numbered from left to right.

row $\times$ col	$D_1$	$D_2$	$D_3$	$D_4$	$D_5$	$D_6$	$D_7$	$D_8$	$D_9$
$D_1$	$pD_1$	$pD_2$	$D_1$	$pD_4$	$D_1$	$pD_6$	$D_2$	$D_4$	$D_6$
$D_2$	$D_1$	$D_2$	$qD_1$	$D_4$	$D_2$	$D_6$	$qD_2$	$qD_4$	$qD_6$
$D_3$	$pD_3$	$pD_7$	$D_3$	$pD_8$	$D_3$	$pD_9$	$D_7$	$D_8$	$D_9$
$D_4$	$pD_4$	$pD_6$	$D_4$	$pqD_4$	$D_4$	$pqD_6$	$D_6$	$qD_4$	$qD_6$
$D_5$	$D_1$	$D_2$	$D_3$	$D_4$	$D_5$	$D_6$	$D_7$	$D_8$	$D_9$
$D_6$	$D_4$	$D_6$	$qD_4$	$qD_4$	$D_6$	$qD_6$	$qD_6$	$q^2D_4$	$q^2D_6$
$D_7$	$D_3$	$D_7$	$qD_3$	$D_8$	$D_7$	$D_9$	$qD_7$	$qD_8$	$qD_9$
$D_8$	$pD_8$	$pD_9$	$D_8$	$pqD_8$	$D_8$	$pqD_9$	$D_9$	$qD_8$	$qD_9$
$D_9$	$D_8$	$D_9$	$qD_8$	$qD_8$	$D_9$	$qD_9$	$qD_9$	$q^2D_8$	$q^2D_9$

We now present a more efficient method of computing multiplication tables which makes use of the bijection established in theorem 20 between  $\mathfrak{sh}$ -classes in  $S_{\min}^3[F, F']$  and those partitions of  $F \dot{\cup} F'$  which are connectivities of concrete diagrams in  $S_{\min}^3[F, F']$ . A convenient way to represent a partition of  $F \dot{\cup} F'$  for this purpose is by means of coloured graphs.

**Definition 27.** Let  $G$  be a graph and  $C$  a set. A colouring of  $G$  by  $C$  is a map from the edge set of  $G$  to  $C$ . Write  $G^C$  for the set of all colourings of  $G$  by  $C$ .

Given  $F$ , let  $\mathcal{G}(F)$  be the rooted undirected graph constructed as follows. The vertex set is the set of connected components of  $\mathbb{R}^2 \setminus F$  (regions), the root being the vertex associated with the unbounded region; there is an edge between two vertices if there is a component in  $F$  which is a boundary between the corresponding regions.

We may associate a rooted tree with the graph  $\mathcal{G}(F)$  by forgetting the labels on all the vertices except the root. Now consider a concrete diagram  $D \in S_{\min}^3[F, F']$  and an injective map  $f$  with domain the set of component of  $D$ . We will say that component  $d$  has ‘colour’  $f(d)$ . Given the pair  $(D, f)$  we define a colouring  $\phi(D, f)$  of the edges of the ordered pair of graphs  $(\mathcal{G}(F), \mathcal{G}(F'))$  as follows: if  $l \in F \dot{\cup} F'$  is in the boundary of component  $d$  in  $D$ , then the colour of the edge associated with  $l$  is  $f(d)$ .

We will regard  $(\mathcal{G}(F), \mathcal{G}(F'))$  as the single tree  $\mathcal{G}(F \dot{\cup} F')$  by identifying the roots. Then  $\phi(D, f)$  defines a partition of the set of edges of  $\mathcal{G}(F \dot{\cup} F')$  which corresponds to the partition  $p(D)$  of the set of loops of  $F \dot{\cup} F'$ . By theorem 20, each such colouring of  $\mathcal{G}(F \dot{\cup} F')$  corresponds to a  $\mathfrak{sh}$ -class in  $S_{\mathfrak{sh}}^3[F, F']$ .

**Definition 28.** For any two edges  $e$  and  $e'$  in a tree let  $ch(e, e')$  be the chain of edges connecting  $e$  to  $e'$  in the tree (excluding  $e$  and  $e'$ ). An element of  $\mathcal{G}(F \dot{\cup} F')^C$  is admissible iff for every pair of same-coloured edges  $e$  and  $e'$ , either there is another edge in  $ch(e, e')$  of the same colour, or else every colour in  $ch(e, e')$  appears an even number of times.

**Proposition 29.** The image under  $\phi$  of the set of  $n$ -component elements of  $S_{\min}^3[F, F']$  is the set of admissible colourings in  $\mathcal{G}(F \dot{\cup} F')^{\{1,2,\dots,n\}}$ .

We refer to [24] for a proof. This proposition establishes a correspondence between admissibly-coloured graphs and  $\mathfrak{sh}$ -classes. Then a  $\mathfrak{h}$ -class can be represented by the coloured graph of any of the  $\mathfrak{sh}$ -classes of which it consists.

The correspondence between  $\mathfrak{h}$ -classes and coloured graphs gives us a practical way of computing compositions of  $\mathfrak{h}$ -classes. Let  $A \in S_{\min}^3[F', F]$  and  $B \in S_{\min}^3[F, F'']$  and  $f$  a map colouring the components of  $A$  and  $B$  so that no colour appears in both  $A$  and  $B$ . To compute  $[A]_{\mathfrak{h}} \cdot [B]_{\mathfrak{h}}$  we first draw the coloured trees  $\phi(A, f)$  and  $\phi(B, f)$ . If the group  $\Pi_F$  has only one element (the identity) there is a well-defined correspondence between the  $F$ -edges of  $\phi(A, f)$  and the  $F$ -edges of  $\phi(B, f)$ . We then identify the colours of every pair of edges that are in correspondence, and propagate this identification to the  $F'$ -edges of  $\phi(A, f)$  and the  $F''$ -edges of  $\phi(B, f)$ . The element of  $\mathcal{G}(F' \dot{\cup} F'')^C$  obtained by joining  $\phi(A, f)$  and  $\phi(B, f)$  at the root after the identification and propagation of colours corresponds to  $[A]_{\mathfrak{h}} \cdot [B]_{\mathfrak{h}}$ . Therefore composition of  $\mathfrak{h}$ -classes coincides with the partition algebra composition of the connectivities if  $\Pi_F$  is trivial. This is not so if the group  $\Pi_F$  has  $n > 1$  elements. Then there are  $n$  ways in which the  $F$ -edges of  $\phi(A, f)$  and the  $F$ -edges of  $\phi(B, f)$  can be put in correspondence. The insertion of a complete set in equation (5.3) corresponds to defining the composition to be the uniformly weighted sum of the  $n$  possible outcomes.

The power of  $q$  in  $[A]_{\mathfrak{h}} \cdot [B]_{\mathfrak{h}}$  is the number of colours after identification which do not propagate to the  $F'$ -edges of  $\phi(A, f)$  or the  $F''$ -edges of  $\phi(B, f)$ . If the number of colours in  $\phi(A, f)$  and  $\phi(B, f)$  before and after identification is, respectively,  $C_b$  and  $C_a$ , then by equation (2.2) the power of  $p$  in  $[A]_{\mathfrak{h}} \cdot [B]_{\mathfrak{h}}$  is  $C_a - C_b + |F|$ .

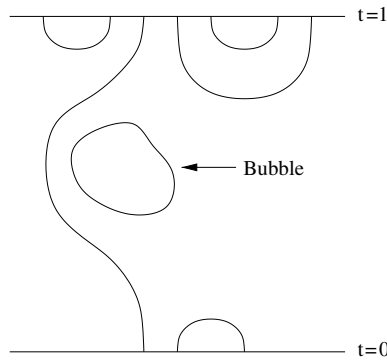
For example, consider again the concrete diagrams shown in figure 2. Note first that  $\Pi_F = 1$ , as the two concentric loops in  $F$  cannot be interchanged by an isotopy. There are nine admissible colourings. Let  $D_1, D_2, \dots, D_9$  be the  $\mathfrak{h}$ -classes defined by those admissible colourings, in the order shown in the right-hand side of figure 3 (colours are indicated by letters r, y, b and g). The composition table for these heterotopy classes is then the one shown at the beginning of this section.

This example is atypical in that the loop configuration  $(F, F)$  has  $\Pi_F = 1$ . Let us now consider the boundary configuration shown in figure 4. There are now two different ways in which the coloured edges of two concrete diagrams can be put in correspondence. Taking the coloured trees shown in figure 4, we find that the two elements of  $\Pi_G$  give rise to two different contributions to the composition  $D_1 \cdot D_1$ , one of which is  $qD_1$  and the other  $D_2$ . Therefore we find that  $D_1 \cdot D_1 = \frac{1}{2}(qD_1 + D_2)$ .

A more detailed analysis of the algebras defined here is given in [24], together with a preliminary analysis of their representation theories, which are intriguingly much richer than the original Temperley–Lieb algebra itself.

## 7. Discussion

In this communication we have constructed, for arbitrary  $d$ , algebras (indeed, categories) whose basis diagrams exist in an ambient space of dimension  $d$ . The particular case  $d = 2$



**Figure 4.** A concrete diagram in  $S^{2,1}$  having a bubble:  $b(D) = 1, g(D) = 0, |D| = 6, \chi(D) = |D| - b(D) = 5$ .

coincides with the Temperley–Lieb algebra already known to play an important role in the analysis of the  $Q$ -state Potts model by transfer matrices, as explained in the introduction.

We will conclude by discussing the relevance of the case  $d = 3$  to the statistical mechanics of three-dimensional lattice models. In particular, we consider the three-dimensional gauge Potts model [18], a simple, discrete forms of the lattice gauge theory which lies at the core of computational particle physics [9, 30]. In this gauge theory, the local states reside at the bonds (also called edges) of a three-dimensional lattice. Transfer matrix layers are essentially two-dimensional, and we will later define a set of ‘gauge’ states which are (after suitable gauge fixing) loops drawn on the bonds of the layer of the lattice. In the continuum limit, these states resemble the ‘boundary configurations’ which we described in the main body of this communication, as we now explain.

In the three-dimensional  $Q$ -state Potts lattice gauge model with lattice edge set  $E$ , the set of gauge field configurations is the set of all functions  $f : E \rightarrow \mathbb{Z}_Q$ . Each configuration gives a field

$$F(x) = e^{2\pi i f(x)/Q}. \tag{7.1}$$

Let  $Y$  and  $BY$  denote respectively a plaquette and its oriented boundary. The orientation of an edge  $x \in BY$  will be denoted  $\epsilon(x)$ , where  $\epsilon(x) = \pm 1$  and  $\sum_{x \in BY} \epsilon(x) = 0$  for any plaquette  $Y$ . Then we define the energy  $H(Y)$  of plaquette  $Y$  and the total energy of the lattice  $H$  to be

$$H(Y) = \frac{1}{Q} \sum_{r=0}^{Q-1} \prod_{x \in BY} F(x)^{r\epsilon(x)}, \quad H = \sum_Y H(Y), \tag{7.2}$$

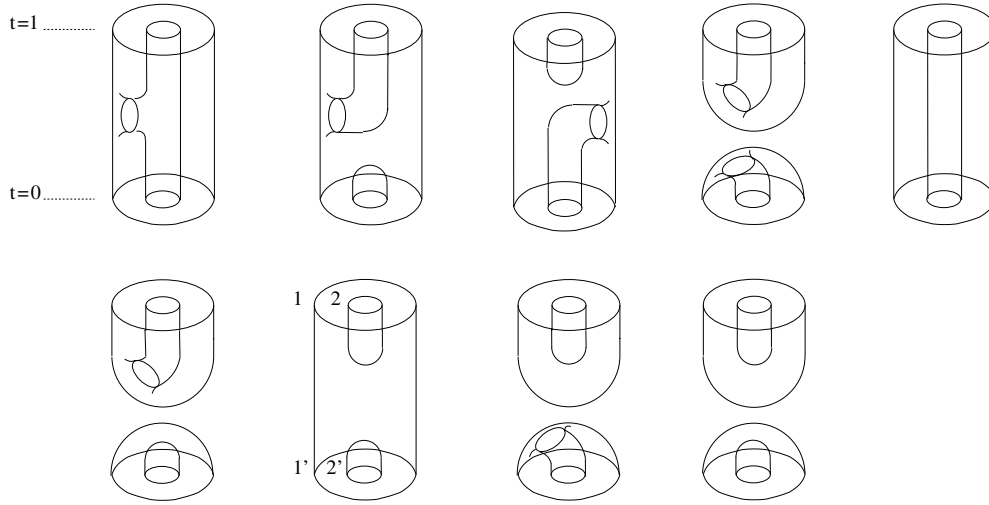
where the sum in  $H$  extends over all the plaquettes in the lattice [18]. Note that  $H(Y)$  is real for all  $Y$ . Indeed, a re-expression, more reminiscent of the Potts Hamiltonian, is

$$H(Y) = \delta_{\sigma_1 \sigma_2 \sigma_3 \sigma_4, 1} \tag{7.3}$$

where we have taken  $BY = \{x_1, x_2, x_3, x_4\}$  and  $\sigma_j = F(x_j)^{\epsilon(x_j)}$ . The partition function of the system is defined in the usual way as  $Z(\beta) = \sum_f e^{-\beta H}$ .

The transfer matrix layer defines a perpendicular direction that we will call ‘timelike’. Bonds and plaquettes perpendicular to this direction are ‘spacelike’. It was shown in [21, equation (10.22)], that, for this model, the transfer matrix on a layer of the lattice can be written as

$$T(v) = \prod_{x \in edge} (v + Q^{1/2} R(U_x)) \prod_{Y \in layer} (1 + v Q^{-1/2} R(U_Y)). \tag{7.4}$$



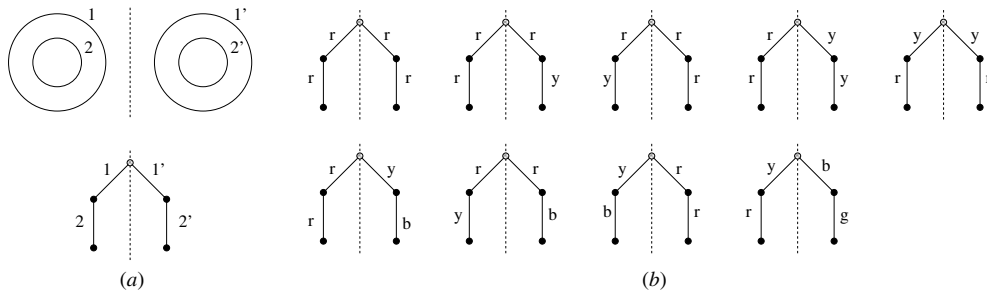
**Figure 5.** Representations of several concrete diagrams in  $S^{3,1}$  whose boundaries are two pairs of concentric circles. One concrete diagram has its boundary loops labelled 1, 2, 1' and 2' for a later purpose.

in which  $U_x$  and  $U_Y$  are operators obeying the relations

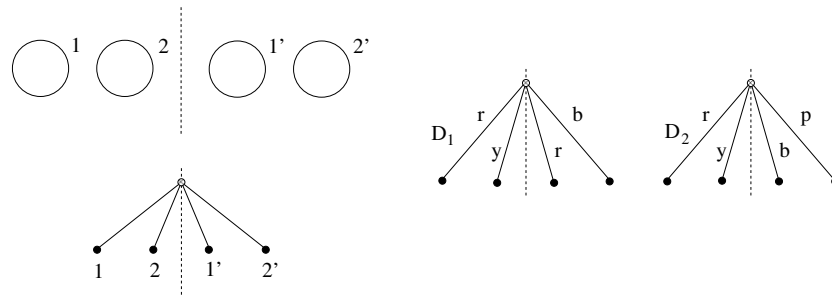
$$\begin{aligned}
 U_x U_x &= Q^{1/2} U_x && \text{if } x \text{ is a spacelike bond} \\
 U_Y U_Y &= Q^{1/2} U_Y && \text{if } Y \text{ is a spacelike plaquette} \\
 U_x U_Y U_x &= U_x && \text{if } x \in BY \\
 U_Y U_x U_Y &= U_Y && \text{if } x \in BY \\
 U_x U_Y &= U_Y U_x && \text{if } x \notin BY \\
 U_x U_{x'} &= U_{x'} U_x && \text{if } x \neq x' \\
 U_Y U_{Y'} &= U_{Y'} U_Y && \text{if } Y \neq Y',
 \end{aligned}$$

and  $R$  is a representation. This is directly analogous to equation (1.2) and, just as we derived a Temperley–Lieb diagram basis for  $D_{2i}$  and  $D_{2i-1}$  in the introduction, so the operators  $U_x$  and  $U_Y$  have a representation in a non-local basis of gauge states, defined as follows. Consider the set of possible closed paths drawn on the bonds of a square lattice, accessible from the empty case (no path) by any sequence of the operations: (i) draw a path around the boundary of a plaquette; (ii) remove a segment common to two paths. Figure 8 shows this construction in a  $3 \times 1$  lattice where the circles represent closed paths of edges. Note that  $v_{14}$  contains two concentric loops, similar to the loop configurations in figures 5 and 6. In this basis, the matrices  $U_x$  and  $U_Y$  are given by  $(U_x)_{ij} = Q^{\delta_{ij}/2}$  if removing all loop segments passing through bond  $x$  (and any open arcs left over) takes state  $v_i$  to state  $v_j$ , and 0 otherwise;  $(U_Y)_{ij} = Q^{\delta_{ij}/2}$  if building a loop around a square  $Y$  takes state  $v_i$  to state  $v_j$ , and 0 otherwise. For example, taking the bond marked  $x$  in state  $v_5$ , we would find that  $U_x v_5 = v_9$  and  $U_x v_2 = U_x v_3 = v_1$ . If plaquettes are numbered as in state  $v_1$ , then, for example,  $U_2 v_2 = v_5$  and  $U_1 v_5 = Q^{1/2} v_5$ .

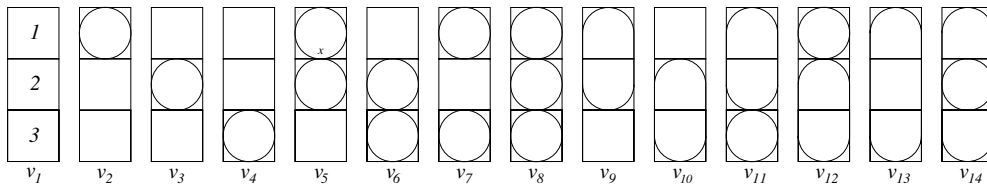
No exact solutions to three-dimensional spin (or dual gauge) lattice models are yet known, despite their importance and a great deal of effort spent on them since their inception [6, 9, 12]. In this context, it is helpful to explore any aspects which *do* generalize to three dimensions, of the contrastingly well-understood two-dimensional models (even if doing so does not lead immediately to a solution of a specific model). We conclude with some remarks in this direction.



**Figure 6.** (a) Construction of  $\mathcal{G}(F \dot{\cup} F)$  from figure 2. (b) Admissible colourings.



**Figure 7.** Boundary configuration  $(G, G)$  with  $\Pi_G = \mathbb{Z}_2$ , and two admissible colourings  $D_1$  and  $D_2$ .



**Figure 8.** States of the gauge Potts model in a  $3 \times 1$  lattice.

The passage from the Hamiltonian (7.3) to the algebra generated by the  $U_x$  and  $U_y$  provides a useful framework in which to analyse the statistical mechanics of the three-dimensional gauge Potts model, analogous to the corresponding two-dimensional case. The algebras defined in the main body of the present communication for  $d = 3$  act on themselves (i.e. in the regular representation) in a way analogous to the way  $U_x$  and  $U_y$  act on the gauge states. This suggests the intriguing possibility of a three-dimensional model associated with the  $d = 3$  algebra in the same way as the classical Hamiltonian (7.3) leads to the  $U_x$  and  $U_y$ . Another interesting possibility would be to construct a three-dimensional classical version of the quantum loop gas in [11], in analogy to the relationship between classical two-dimensional lattice models and one-dimensional quantum spin chains [5]. In order to do this, it would be useful to have a generalization of the tensor space representation of Temperley–Lieb on which spin chains act. It is not yet clear what form such generalization would take.

**References**

[1] Adamovich A M 1992 Composition factors and submodules of Weyl modules *PhD Thesis* Moscow State University



- [2] Armstrong M A 1979 *Basic Topology* (New York: McGraw-Hill)
- [3] Baez J and Dolan J 1995 Higher-dimensional algebra and topological quantum field theory *J. Math. Phys.* **36** 6073–105
- [4] Baxter R J 1973 *J. Stat. Phys.* **8** 25
- [5] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (New York: Academic)
- [6] Baxter R J and Bazhanov V V 1992 New solvable lattice models in three-dimensions *J. Stat. Phys.* **69** 453–585
- [7] Bazhanov V V and Sergeev S M 2006 Zamolodchikov’s tetrahedron equation and hidden structure of quantum groups *J. Phys. A: Math. Gen.* **39** 3295
- [8] Blote H W and Nightingale M P 1982 Critical behaviour of the two-dimensional Potts model with a continuous number of states: a finite size scaling analysis *Physica A* **112** 405–65
- [9] Creutz M 1985 *Quarks, Gluons and Lattices (Cambridge Monographs on Mathematical Physics)* (Cambridge: Cambridge University Press)
- [10] Dasmahapatra S and Martin P P 1996 On the algebraic approach to cubic lattice Potts model *J. Phys. A: Math. Gen.* **29** 263–78
- [11] Freedman M H A magnetic model with a possible Chern–Simons phase *Preprint quant-ph/0110060*
- [12] Ising E 1925 Contribution to the Theory of Ferromagnetism *Z. Phys.* **31** 253–8
- [13] Jimbo M 1985 A  $q$  difference analog of  $U(g)$  and the Yang–Baxter equation *Lett. Math. Phys.* **10** 63–9
- [14] Kashaev R M, Mangazeev V V and Stroganov Yu G 1993 Star square and tetrahedron equations in the Baxter–bazhanov model *Int. J. Mod. Phys. A* **8** 1399–410
- [15] Kassel C 1995 *Quantum Groups* (Berlin: Springer)
- [16] Kaufman L 1993 *Knots and Physics* (Singapore: World Scientific)
- [17] Khovanov M 1999 A categorification of the Jones polynomial *Preprint math/9908171*
- [18] Khovanov M and Seidel P 2000 Quivers, Floer cohomology, and braid group actions *Preprint math/0006056*
- [19] Kogut J B 1979 An introduction to lattice gauge theory and spin systems *Rev. Mod. Phys.* **51** 659–713
- [20] Koo W M T and Saleur H 1994 Representations of the Virasoro algebra from lattice models *Nucl. Phys. B* **426** 459–504
- [21] Laures G 2000 On cobordism of manifolds with corners *Trans. Am. Math. Soc.* **352** 5667–88
- [22] Martin P P 1991 *Potts Models and Related Problems in Statistical Mechanics* (Singapore: World Scientific)
- [23] Martin P P 1994 Temperley–Lieb algebras for non-planar statistical mechanics—the partition algebra construction *J. Knot Theory Ramifications* **3** 51–82
- [24] Martin P P 2000 The partition algebra and the Potts model transfer matrix spectrum in high dimensions *J. Phys. A: Math. Gen.* **32** 3669–95
- [25] Martin P P and Alvarez M A Temperley–Lieb category for 2-manifolds, in preparation
- [26] Martin P P and Saleur H 1993 On an algebraic approach to higher dimensional Statistical Mechanics *Commun. Math. Phys.* **158** 155–90
- [27] Mendelson B 1962 *Introduction to Topology* (Glasgow: Blackie)
- [28] Milnor W 1963 *Morse Theory* (Princeton, NJ: Princeton University Press)
- [29] Moise E E 1977 *Geometric Topology in Dimensions 2 and 3* (Berlin: Springer)
- [30] Temperley H N V and Lieb E H 1971 Relations between the ‘percolation’ and ‘colouring’ problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the ‘percolation’ problem *Proc. R. Soc. London A* **322** 251–80
- [31] Wilson K G 1974 Confinement of Quarks *Phys. Rev. D* **10** 2445–59